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Infinite Hierarchy of Exponents in a Two-Component Random Resistor Network

Lucilla de Arcangelis[1,2] **and Antonio Coniglio**[3]

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We have studied the voltage distribution for a two-component random mixture of conductances σ_a and σ_b. A scaling theory is developed for the moments of the distribution, which predicts, for small values of $h=\sigma_a/\sigma_b$, an infinite number of crossover exponents, one for each moment, for Euclidean dimension $d>2$, and only one crossover exponent for $d=2$. Monte Carlo results on the square lattice confirm this prediction.

KEY WORDS: Percolation; two-component mixture; scaling theory; crossover exponent; voltage distribution; multifractality.

Recently the study of the voltage distribution in a random resistor network[1-4] has given rise to much interest due to its peculiar behavior. It was found that the moments of the voltage distribution at the percolation threshold scale with an infinite number of independent exponents as the system size L goes to infinity. More recently this infinite hierarchy of exponents was related to the multifractal structure of the incipient infinite cluster and a similar behavior was also found in a random superconducting network.[2] In this paper we consider the general case of the voltage distribution of a two-component random resistor network. To be more precise, we consider a d-dimensional hypercubic lattice and we associate to each bond ij a conductance $\sigma_{ij}=\sigma_a$ with probability p and $\sigma_{ij}=\sigma_b$ with probability $1-p$. If a unit voltage $\Delta V=1$ is applied between the two opposite edges of the network, each bond will be characterized by a voltage

[1] Institute for Theoretical Physics, University of Cologne, 5000 Köln 41, West Germany.
[2] Present address: Service de Physique Théorique, CEN, Saclay, 91191 Gif sur Yvette, France.
[3] Dipartimento di Fisica, Università di Napoli, 80125 Napoli, Italy.

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drop with absolute value V_{ij} and power dissipated $\sigma_{ij} V_{ij}^2$. Next consider the moments of the power dissipated

$$M(k, \sigma_a, \sigma_b) = \left\langle \sum_{ij} (\sigma_{ij} V_{ij}^2)^{k/2} \right\rangle \tag{1}$$

where the brackets denote the average over all bond configurations.

At $p = p_c$ two limiting cases have already been studied: (i) the random resistor network, $\sigma_a = 1$ and $\sigma_b = 0$, in which the moments scale as

$$M(k, 1, 0) \sim L^{-\tilde{p}(k)} \tag{2}$$

and (ii) the random superconducting network ($\sigma_a = \infty$ and $\sigma_b = 1$). In this case $M(k, \infty, 1)$ is made of two contributions: a divergent one, arising from the spanning configurations of superconducting bonds, and a finite one, $M'(k, \infty, 1)$, coming from the nonspanning configurations, which scales as

$$M'(k, \infty, 1) \sim L^{\tilde{\xi}'(k)} \tag{3}$$

Note that in these limits (2) and (3) coincide with the kth moments of the voltage distribution.

Since the voltage distribution is invariant under the transformation $\sigma_{ij} \to \lambda \sigma_{ij}$, from (1) we obtain the following homogeneity relation:

$$M(k, \lambda \sigma_a, \lambda \sigma_b) = \lambda^{k/2} M(k, \sigma_a, \sigma_b) \tag{4}$$

Choosing first $\lambda = 1/\sigma_a$ and then $\lambda = 1/\sigma_b$, we find the following relation

$$M(k, 1, h) = h^{k/2} M(k, h^{-1}, 1) \tag{5}$$

where $h = \sigma_b/\sigma_a$.

Note that, in the limit $h \to 0$, the left-hand side approaches $M(k, 1, 0)$, the moments in the random resistor network, and the right-hand side approaches $h^{k/2} M(k, \infty, 1)$, the moments in the random superconducting network. Since for $k = 2$ the moment coincides with the conductance, Eq. (5) reduces to an already known result, relating the conductances of a random resistor and a random superconducting network.[5]

We consider now the case $h \ll 1$. The variable h plays the same role as a magnetic field in an Ising model,[5,6] i.e., when h is different from zero the critical behavior disappears. Therefore, the natural scaling ansatz is

$$M(k, 1, h) \sim h^u [f_1(k, Lh^\varphi) + f_2(k, Lh^\varphi)] \tag{6a}$$

and from (5)

$$M(k, h^{-1}, 1) \sim h^{u - k/2} [f_1(k, Lh^\varphi) + f_2(k, Lh^\varphi)] \tag{7a}$$

where f_1 and f_2 are the contributions, respectively, from the spanning and the nonspanning configurations; for small values of h, f_2 is neglegible with respect to f_1 in (6a). On the other hand, in (7a) the first term is the divergent one and only f_2 contributes to $M'(k, h^{-1}, 1)$. Therefore we can write

$$M(k, 1, h) = h^u f_1(k, Lh^\varphi) \tag{6b}$$

and from (5)

$$M'(k, h^{-1}, 1) = h^{u-k/2} f_2(k, Lh^\varphi) \tag{7b}$$

For $Lh^\varphi \ll 1$, (6b) and (7b) must behave, respectively as (2) and (3). Consequently, we have that the functions $f_1(k, x) \sim x^{-u/\varphi} g_1(k, x)$ and $f_2(k, x) \sim x^{-(u-k/2)/\varphi} g_2(k, x)$, with the exponents u and φ given by

$$u = (k/2)\,\tilde{p}(k)/[\tilde{\xi}'(k) + \tilde{p}(k)] \tag{8}$$

and

$$\varphi = k/2[\tilde{\xi}'(k) + \tilde{p}(k)] \tag{9}$$

and $g_1(k, 0)$ and $g_2(k, 0)$ constants. In this limit (6b) and (7b) are written as

$$M(k, 1, h) = L^{-\tilde{p}(k)} g_1(k, Lh^\varphi) \tag{10}$$

$$M'(k, h^{-1}, 1) = L^{\tilde{\xi}'(k)} g_2(k, Lh^\varphi) \tag{11}$$

For the particular case $d = 2$, duality arguments give $\tilde{\xi}'(k) = k\tilde{p}(2) - \tilde{p}(k)$; therefore, for every value of k, $u = \tilde{p}(k)/2\tilde{p}(2)$ and $\varphi = 1/2\tilde{p}(2)$, independent of k, where $\tilde{p}(2) \approx 0.98$ is the conductivity exponent, which has been evaluated numerically very accurately (see Ref. 2 and references therein).

In order to test the scaling prediction (10), we have performed numerical simulations on an $L \times L$ square lattice at the percolation threshold ($p = p_c = 0.5$), where to each bond is assigned a conductance equal to 1 with probability p and a conductance equal to h with probability $1 - p$. We have analyzed system sizes $L = 40$, 60, 80, 100 for different values of h, $h = 0.1$, 0.05, 0.01, 0.001, and averaged our data over many configurations (2000, 1000, 300, 200 configurations, respectively, for $L = 40$, 60, 80, 100 at $h = 0.001$).

A unit potential drop is imposed across the sample and free boundary conditions are used in the transverse direction. Only those configurations are analyzed in which a spanning cluster of bonds with unit conductance is present. For each spanning configuration we then calculate the voltage at each node of the network by a standard numerical relaxation algorithm.

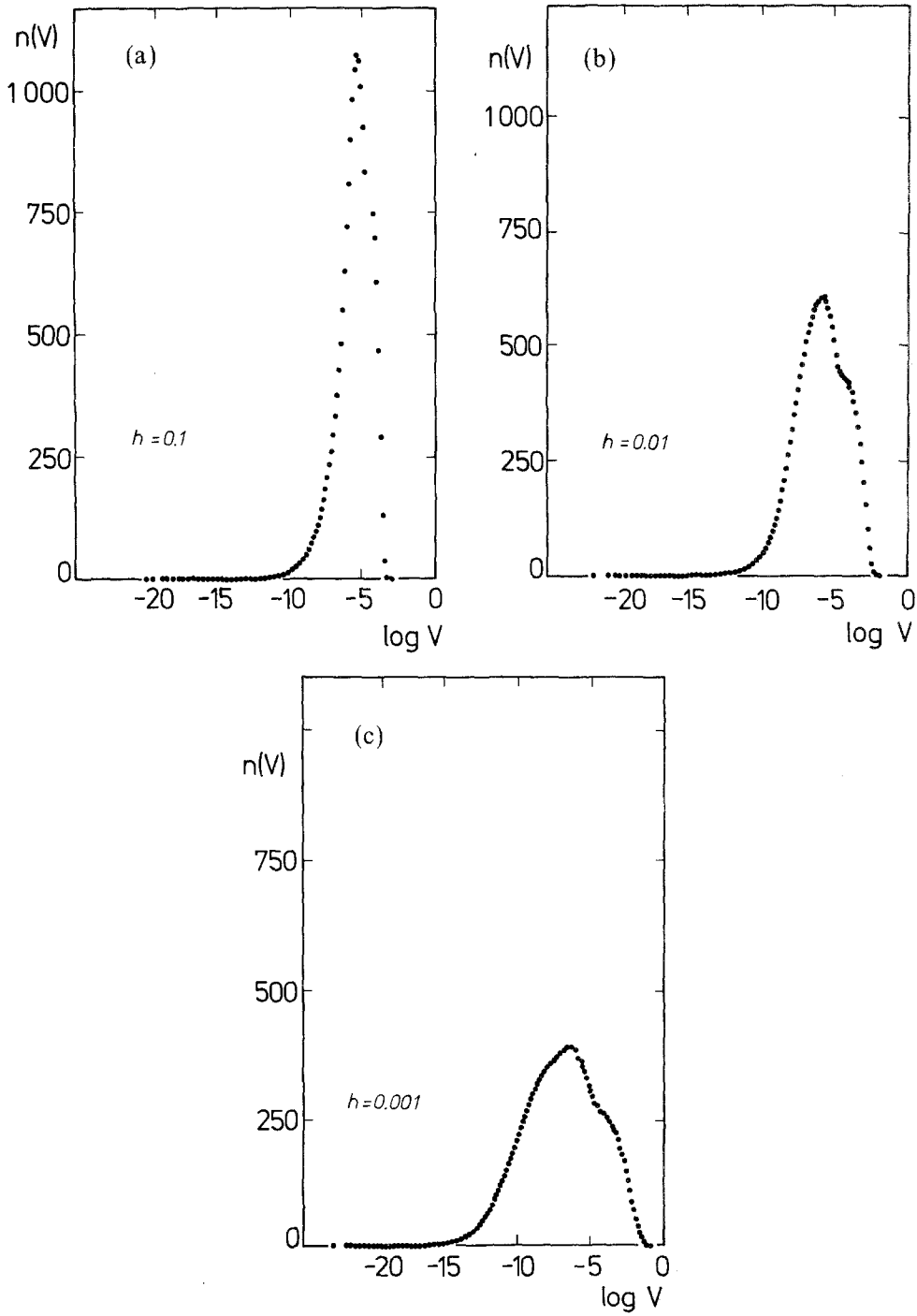


Fig. 1. The average voltage distribution of a 100×100 square lattice at p_c plotted versus $\log V$ for (a) $h = 0.1$ (500 configurations), (b) $h = 0.01$ (300 configurations), and (c) $h = 0.001$ (200 configurations). The average is taken over the spanning configurations.

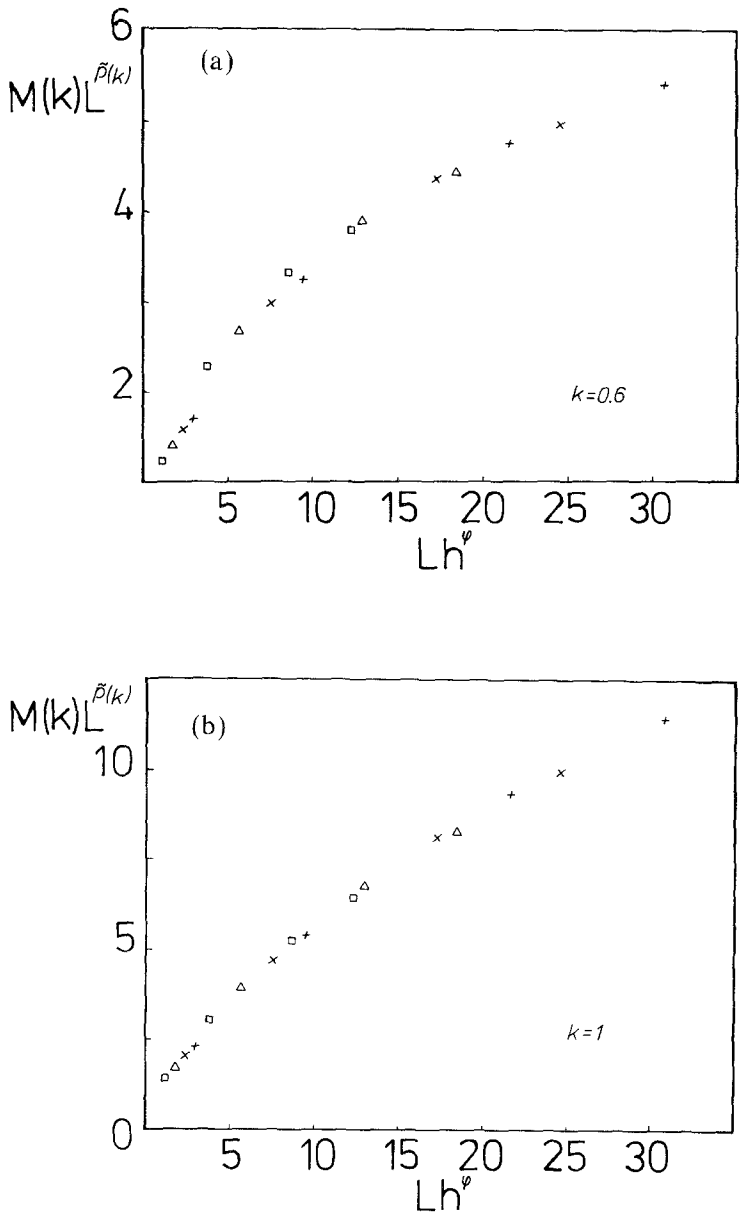
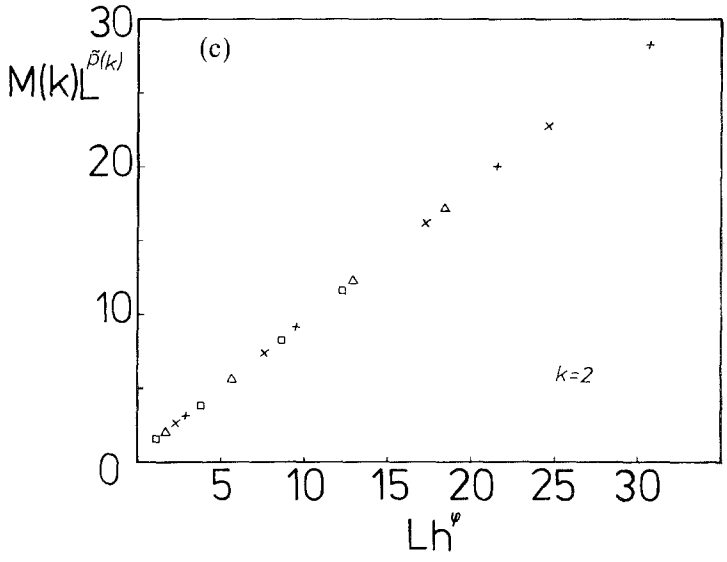


Fig. 2. The quantity $g_1(k, Lh^{\varphi}) = M(k, 1, h)\, L^{\tilde{p}(k)}$ plotted as a function of Lh^{φ} for (a) $k = 0.6$, (b) $k = 1$, (c) $k = 2$, (d) $k = 3$, and (e) $k = 4$. For each value of k, the data corresponding to different system sizes L(□, 40; △, 60; ×, 80; +, 100) and different values of h collapse on the same universal curve.



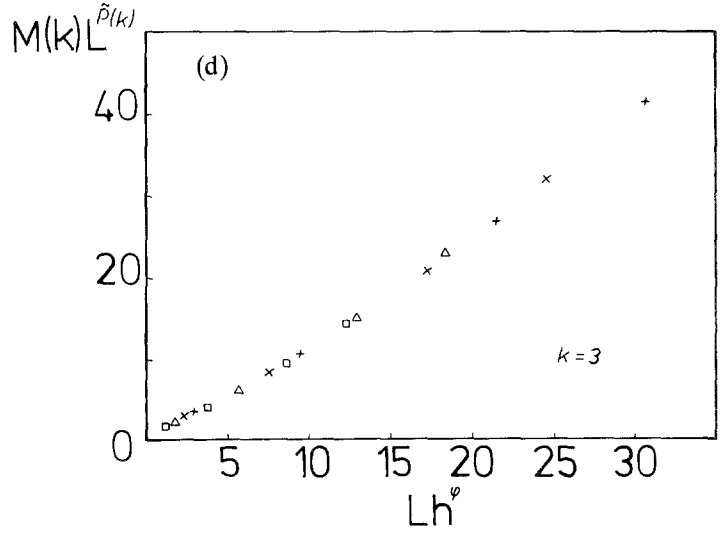


Fig. 2 (*continued*)

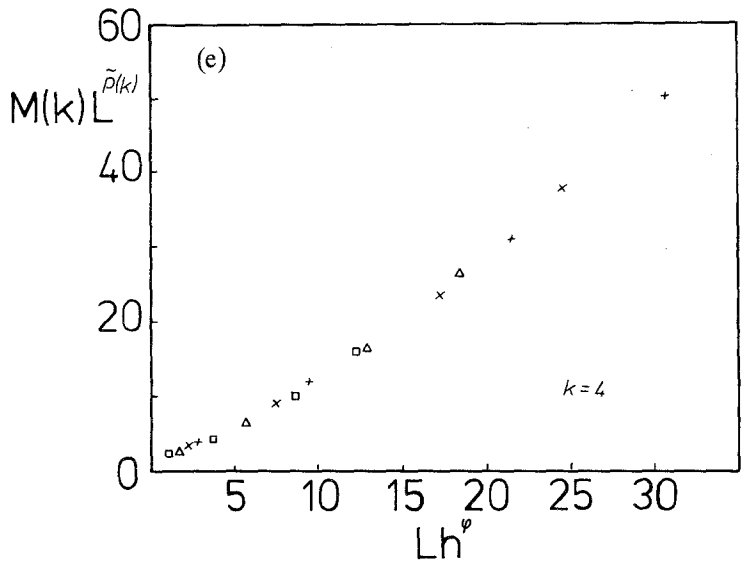


Fig. 2 (*continued*)

First we analyze the voltage distribution $n(V)$ for different values of h (Fig. 1). The qualitative behavior of $n(V)$ is as expected: for $h=1$, that is, for an homogeneous network of unit conductances, the voltage distribution is a delta function peaked at a value of the voltage equal to $1/L$. For h still close to 1 ($h=0.1$), the voltage distribution $n(V)$ is a sharply peaked function that maintains the main features of a delta function. As the value of h decreases, the distribution widens over a larger range of V values, exhibiting the characteristic low-voltage tail present in the voltage distribution of the random resistor network.

Then we test the scaling prediction (10). For different moments of the power dissipated we consider the scaling function $g_1(k, Lh^\varphi)$, which, for each value of k, is expected to be a function of the variable Lh^φ, where the crossover exponent is $\varphi=0.51$. In Fig. 2 the scaling functions $g_1(k, Lh^\varphi)=M(k, 1, h)\, L^{\tilde{p}(k)}$ are plotted as function of Lh^φ for the whole range of L and h values and for different values of k, where the values of $\tilde{p}(k)$ have been taken from the literature.[2] The collapse of our data fully confirms (10). Note also that our data show that $g_1(k, Lh^\varphi)$ is a convex function of Lh^φ for $k>1$ and a concave function for $k<1$.

In conclusion, we have found that the crossover exponents of the various moments of the power dissipated for a two-component random mixture in two dimensions are characterized by one single exponent. It

would be interesting to study the problem further in higher dimensionalities, where an infinite number of crossover exponents is predicted.

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